Unsteady, viscous, circular flow

I. The line impulse of angular momentum

By MERWIN SIBULKIN
Convair Scientific Research Laboratory, San Diego

(Received 27 December 1960)

In this paper a study of the energy-transfer processes associated with the motion of a viscous, heat-conducting fluid is begun. The class of motions considered are unsteady, two-dimensional, vortical flows. After developing simplified equations of motion and energy appropriate to this type of flow in the low Mach-number limit, general solutions of the momentum equations are presented. The concept of a *line impulse of angular momentum* is introduced as an example of this class of motions for which a solution of the energy field is obtainable in closed form. The solution for the line impulse can be viewed as a combination of velocity, pressure, and temperature waves concurrently radiating from the origin of the impulse and decaying with time. Particular examples of the development of the energy field of the impulse in both liquids and gases are presented for selected values of Prandtl number. The energy-transfer processes are discussed in some detail, and the resulting differences in the energy fields for liquid gases are emphasized.

1. Introduction

For a large class of problems in hydrodynamics and low-speed aerodynamics, it is not necessary to study the energy equation to obtain the desired solution of the flow problem. In inviscid, compressible flow the energy equation *per se* is eliminated by the use of the isentropic relationship $p/\rho^\gamma = \text{const}$. And, for certain viscous, compressible flow problems the energy equation is satisfied for particular values of the Prandtl number σ by a constant value of the stagnation temperature throughout the flow field, e.g. the flow over an insulated plate with $\sigma = 1$ and the flow through a shock wave with $\sigma = \frac{3}{4}$. Consequently, there have been relatively few problems in fluid dynamics in which solutions of the energy equation have been obtained compared to those in which the momentum equation has been solved.

In this investigation, interest will be focused upon the transfer of energy between fluid elements due to viscous work and to heat conduction within (but not across) the boundaries of the fluid. The particular class of problems studied are unsteady, two-dimensional flows with circular streamlines. This work was oiiginally motivated by a desire to understand the temperature separation phenomenon exhibited by the Ranque–Hilsch vortex tube.

The investigation is divided into three parts.

In part I the energy field in an infinite fluid associated with the decay of what will be defined as a *line impulse of angular momentum* is considered. The flows of both a liquid and of a gas in the limit of Mach number equal to zero are studied, and solutions, in closed form, are obtained and compared.

The investigation will proceed as follows. First, simplified forms of the Navier–Stokes and energy equations for plane, axisymmetric flow are developed for the case where the Mach number in the flow field is everywhere much less than one. Then the general solutions for the decay of the velocity field associated with an arbitrary initial velocity distribution are given for finite and for infinite flow fields.

The particular initial velocity distribution for a *line impulse of angular momentum* is then introduced. The line impulse of angular momentum may be associated with the following physical model. If a cylindrical rod whose radius is very small compared to its length is immersed in a viscous fluid and impulsively set in rotation about its axis, the adjacent fluid will also be set in motion. The line impulse of angular momentum corresponds to the limit obtained as the radius of the rod approaches zero and the angular velocity of the rod approaches infinity in such a way that the angular momentum imparted to the fluid remains finite.

The solution obtained for the velocity field associated with the decay of the line impulse is then used to determine the energy dissipation due to viscosity. Finally, the energy fields resulting from the combined action of dissipation and heat conduction are found for both incompressible and compressible fluids.

In part II the corresponding flows in a circular cylinder of finite radius will be considered, and the energy transfer processes will be discussed in some detail. In these cases it will be necessary to resort to numerical integration to obtain the desired solutions of the energy equation.

In part III a new model for the flow in a vortex tube will be proposed. Using the methods developed in part II, velocity and energy profiles will be calculated and compared with previously published measurements. Some new experimental work based upon the proposed flow model will also be presented.

2. Notation

Because of the large number of symbols required, it has been convenient in a few instances to assign more than one meaning to a symbol where, it is hoped, no confusion should result.

The term *total* denotes the addition of the kinetic energy per unit mass to the specified quantity; the term *over-all* denotes an integrated value (e.g. of kinetic energy) for a disk of fluid of unit depth. An overbar indicates a mean value, and an asterisk indicates a non-dimensional quantity. Finally, a subscript ∞ denotes the value taken as $r \to \infty$.

3. Preliminary analysis of the conservation equations

In the present work we shall consider two idealized fluids: (i) the *perfect gas* having the equation of state $p = \rho\mathscr{R}T$ and constant values of specific heat c_p and c_v, and (ii) the *perfect liquid* having the equation of state $\rho =$ const. and a single, constant specific heat c. Hereafter, when the terms *gas* or *liquid* are used they will be understood to refer to the perfect fluids defined above.

In cylindical polar co-ordinates (r, θ, z) the continuity, momentum, and energy equations for two-dimensional $(\partial/\partial z = 0)$, axisymmetric $(\partial/\partial\theta = 0)$ flow are (see, for example, Pai 1956, pp. 28–44, and Howarth 1953, pp. 38–54):

$$\frac{\partial\rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(r\rho u) = 0, \tag{3.1a}$$

$$\rho\left(\frac{Du}{Dt}-\frac{v^2}{r}\right) = \frac{1}{r}\frac{\partial(r\sigma_{rr})}{\partial r}-\frac{\sigma_{\theta\theta}}{r}, \tag{3.1b}$$

$$\rho\left(\frac{Dv}{Dt}+\frac{uv}{r}\right) = \frac{1}{r}\frac{\partial(r\tau_{r\theta})}{\partial r}+\frac{\tau_{r\theta}}{r}, \tag{3.1c}$$

$$\rho c_p\frac{DT}{Dt} = -\frac{T}{\rho}\left(\frac{\partial p}{\partial T}\right)_p\frac{Dp}{Dt}+\Phi+\frac{1}{r}\frac{\partial}{\partial r}\left(rk\frac{\partial T}{\partial r}\right), \tag{3.1d}$$

where t is the time, ρ is the fluid density, u the radial velocity, v the circumferential velocity, p the pressure, and T the temperature in the fluid, and where

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t}+u\frac{\partial}{\partial r}, \quad \sigma_{rr} \equiv -p+2\mu\frac{\partial u}{\partial r}+\frac{\lambda}{r}\frac{\partial(ru)}{\partial r}, \quad \sigma_{\theta\theta} \equiv -p+2\mu\frac{u}{r},$$

$$\tau_{r\theta} \equiv \mu\left(\frac{\partial v}{\partial r}-\frac{v}{r}\right), \quad \Phi \equiv \mu\left[2\left(\frac{\partial u}{\partial r}\right)^2+2\left(\frac{u}{r}\right)^2+\left(\frac{\partial v}{\partial r}-\frac{v}{r}\right)^2\right]+\lambda\left(\frac{\partial u}{\partial r}+\frac{u}{r}\right)^2,$$

μ being the first, λ the second, coefficient of viscosity, and k being the thermal conductivity.

We now consider the relative magnitudes of u and v. For a source-free liquid, the continuity equation (3.1 a) immediately gives $u = 0$. To study the situation for a gas we choose a characteristic radius R and velocity V and define the non-dimensional variables $s \equiv r/R$ and $\tau \equiv tV/R$. In the limit $M \to 0$ where M is the Mach number (in the sense $V \to 0$ for finite T), the changes in fluid properties will be small perturbations about the mean values $\bar{p}, \bar{\rho}, \bar{T}, \bar{\mu}$, and $\bar{k}$. The continuity equation (3.1 a) can be formally integrated in the form

$$\frac{u}{V} = -\frac{1}{(\rho/\bar{\rho})\,s}\int_0^s \frac{1}{\bar{\rho}}\frac{\partial\rho}{\partial\tau}\,s'\,ds'. \tag{3.2}$$

Defining the non-dimensional variables

$$h^* \equiv \frac{h-\bar{h}}{\frac{1}{2}V^2}, \quad p^* \equiv \frac{p-\bar{p}}{\frac{1}{2}\bar{\rho}V^2}, \quad \rho^* \equiv \frac{\rho-\bar{\rho}}{\mathscr{M}\bar{\rho}}; \quad \mathscr{M} \equiv \frac{V^2}{2\bar{h}} = \frac{\gamma-1}{2}M^2, \tag{3.3}$$

where h is the enthalpy per unit mass, and γ the ratio of the specific heats, the gas law can be written in the non-dimensional form

$$\rho^* = \{\gamma/(\gamma-1)\}p^*-h^*. \tag{3.4}$$

Differentiating (3.4) and substituting the result in (3.2) gives

$$\frac{u}{V} = -\frac{\mathscr{M}}{(\rho/\bar{\rho})\,s}\int_0^s\left[\left(\frac{\gamma}{\gamma-1}\right)\frac{\partial p^*}{\partial\tau}-\frac{\partial h^*}{\partial\tau}\right]s'\,ds'. \tag{3.5}$$

Since we consider only those changes in enthalpy due to viscous work (by restricting the analysis to flows with no heat transfer across the boundaries of the fluid), h^* remains finite as $\mathscr{M} \to 0$ (as, of course, does p^*), and therefore

$$\lim_{\mathscr{M}\to 0} \frac{u}{V} = 0. \tag{3.6}$$

For estimation purposes (again considering the limit $\mathscr{M} \to 0$), one may take $\mu/\bar{\mu} = h/\bar{h}$ or

$$\frac{\mu-\bar{\mu}}{\bar{\mu}} = \frac{h-\bar{h}}{\bar{h}} \simeq \frac{V^2}{2\bar{h}} \equiv \mathscr{M}.$$

Then
$$\lim_{\mathscr{M}\to 0} \frac{\partial}{\partial r}\left(\mu r \frac{\partial v}{\partial r}\right) = \lim_{\mathscr{M}\to 0} \bar{\mu}\frac{\partial}{\partial r}\left\{[1+O(\mathscr{M})]\, r\frac{\partial v}{\partial r}\right\} = \bar{\mu}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right), \tag{3.7}$$

and similarly for the heat conduction term in the energy equation.

Applying (3.6) and (3.7) to (3.1) eliminates the continuity equation and reduces the momentum and energy equations to

$$\frac{\bar{\rho} v^2}{r} = \frac{\partial p}{\partial r}, \tag{3.8a}$$

$$\bar{\rho}\frac{\partial v}{\partial t} = \frac{\bar{\mu}}{r^2}\frac{\partial}{\partial r}\left[r^2\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right)\right], \tag{3.8b}$$

$$\text{(liquid)} \quad \bar{\rho}\frac{\partial e}{\partial t} = \bar{\mu}\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right)^2 + \frac{\bar{k}}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right), \tag{3.8c}$$

$$\text{(gas)} \quad \bar{\rho}\frac{\partial h}{\partial t} = \frac{\partial p}{\partial t} + \bar{\mu}\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right)^2 + \frac{\bar{k}}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right), \tag{3.8d}$$

where e is the internal energy per unit mass.

Comparing (3.8) to the more familiar steady, boundary-layer equations we note that the momentum equations for a liquid and for a gas become identical and are uncoupled from the energy equation as in the boundary-layer case. However, the energy equations for a liquid and for a gas differ in the unsteady terms.

The unsteady pressure term in (3.8d) can be eliminated by the use of the radial momentum equation (3.8a). After integrating (3.8a) with respect to r and differentiating with respect to t, the resulting expression in non-dimensional variables is

$$\frac{\partial p^*}{\partial \tau} = \frac{\partial p^*(a)}{\partial \tau} + 2\int_{a/R}^{s}\left(\frac{\rho}{\bar{\rho}}\frac{\partial (v^*)^2}{\partial \tau} + \mathscr{M}\frac{\partial \rho^*}{\partial \tau}\right)\frac{ds'}{s'},$$

where a is an arbitrary fixed point in the fluid. Thus

$$\lim_{\mathscr{M}\to 0} \frac{\partial p^*}{\partial \tau} = \frac{\partial p^*(a)}{\partial \tau} + 2\int_{a/R}^{s} \frac{\partial (v^*)^2}{\partial \tau}\frac{ds'}{s'} \tag{3.9}$$

and, it may be noted, $v^*(s, \tau)$ may be found from the circumferential momentum equation (3.8b) above.

The remainder of the analysis is restricted to the limit $M \to 0$; the bars over the fluid properties are no longer needed and will be omitted.

A further mathematical simplification of equations (3.8) can be achieved by employing the angular velocity ω ($= v/r$) as a dependent variable. In terms of ω we obtain

$$\rho r\omega^2 = \partial p/\partial r, \tag{3.10a}$$

$$\frac{\partial\omega}{\partial t} = \frac{\nu}{r^3}\frac{\partial}{\partial r}\left(r^3\frac{\partial\omega}{\partial r}\right), \tag{3.10b}$$

$$\rho c_p\frac{\partial T}{\partial t} = \frac{\partial p}{\partial t}\delta_{fg} + \mu r^2\left(\frac{\partial\omega}{\partial r}\right)^2 + \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right), \tag{3.10c}$$

where ν is the kinematic viscosity and (3.8*c*) and (3.8*d*) have been combined into (3.10*c*), for the sake of economy, by using the 'Kronecker δ' defined below:

for fluid = liquid, $\delta_{fg} = 0$, $c_p dT = c\,dT = de$,

and for fluid = gas, $\delta_{fg} = 1$, $c_p dT = dh$.

Finally, since we are interested in the total-energy field, we combine the momentum equation (multiplied by ω) with the energy equation to obtain the 'total-energy equation'

$$\frac{\partial\Xi}{\partial t} - \frac{\nu}{\sigma r}\frac{\partial}{\partial r}\left(r\frac{\partial\Xi}{\partial r}\right) = \frac{1}{\rho}\frac{\partial p}{\partial t}\delta_{fg} + \frac{\nu}{\sigma r}\frac{\partial}{\partial r}\left[\frac{(\sigma-1)\,r^3}{2}\frac{\partial\omega^2}{\partial r} - r^2\omega^2\right], \tag{3.11}$$

where, anticipating the application of (3.11) to an infinite flow field we have defined a general-purpose total energy (or enthalpy) variable Ξ by

$$\Xi \equiv c_p T + \tfrac{1}{2}v^2 - c_p T_\infty;$$

thus $\Xi = E - e_\infty$ (for a liquid), $\quad \Xi = H - h_\infty$ (for a gas). (3.11*a*)

Here E is the 'total internal energy' and H the total enthalpy, each being obtained by adding the kinetic energy to the previously defined quantities. The circumferential momentum and total-energy equations in the form (3.10*b*) and (3.11) form the basis of the subsequent analysis.

4. Solution of the momentum equation

4.1. *General solutions*

The boundary conditions for the circumferential momentum equation (3.10*b*) are determined by the requirements that $v = 0$ at $r = 0$ and, for an infinite fluid, $v = 0$ at $r = \infty$. When the fluid is bounded by a fixed cylinder of finite radius R,† $v = 0$ at $r = R$. In terms of $\omega(r,t)$, we have the relations

$$(\partial\omega/\partial r)(0,t) = 0, \tag{4.1a}$$

$$\omega(R,t) = 0 \quad \text{or} \quad \omega(\infty,t) = 0. \tag{4.1b}$$

The separation of variables technique and (4.1*a*) give as a solution of (3.10*b*)

$$\omega(r,t) = (A/r)\,e^{-\alpha t}J_1(\alpha r^2/\nu)^{\frac{1}{2}}, \tag{4.2}$$

where J_1 is the Bessel function of the first kind and A and α are constants to be determined.

† It should be clear that this use of the symbol R is not inconsistent with the use of R as a characteristic radius in §3.

For the cylinder of finite radius, boundary condition (4.1 *b*) gives $\alpha_i = \lambda_i^2 \nu / R^2$ where the λ_i are the zeros of J_1. Then, expanding an arbitrary initial condition $\omega(r, 0)$ in a Fourier–Bessel series in J_1 and combining the result with (4.2) gives the general solution (McLeod 1922)

$$v(r,t) = \sum_{i=1}^{\infty} 2R \exp\left(-\frac{\lambda_i^2 \nu t}{R^2}\right) \frac{J_1(\lambda_i s)}{[J_0(\lambda_i)]^2} \int_0^1 s^2 \omega(sR, 0)\, J_1(\lambda_i s)\, ds, \tag{4.3}$$

where $s \equiv r/R$.

For the infinite fluid, setting $k \equiv (\alpha/\nu)^{\frac{1}{2}}$ and applying the Fourier–Bessel transform to (4.2) yields

$$v(r,t) = \int_0^{\infty} \left[\int_0^{\infty} J_1(kr')\, J_1(kr)\, e^{-k^2 \nu t}\, k\, dk\right] r'^2\, \omega(r', 0)\, dr', \tag{4.4}$$

which may be integrated over k (using 4.14 (39) of Erdelyi, Magnus, Oberhettinger & Tricomi 1954), to give the general solution

$$v(r,t) = \int_0^{\infty} \frac{r'^2 \omega(r', 0)}{2\nu t} \exp\left(-\frac{r^2 + r'^2}{4\nu t}\right) I_1\left(\frac{rr'}{2\nu t}\right) dr', \tag{4.5}$$

where I_1 is the modified Bessel function of the first kind.

4.2. *Velocity field of the line impulse of angular momentum*

We define the *cylindrical impulse of angular momentum* $\Omega(r')$ as the δ-function of ω which is everywhere zero except at $r = r'$, where it tends to infinity in such a way that

$$2\pi \int_0^{\infty} \omega(r)\, r^3\, dr \equiv \Omega(r'). \tag{4.6}$$

For $r' = 0$, equation (4.6) defines the *line impulse of angular momentum* which will be designated simply by Ω.

Combining (4.6) with (4.5), the time-dependent velocity field of a cylindrical impulse occurring at $t = 0$ is given by

$$v(r,t;r') = \frac{r\Omega(r')}{8\pi\nu^2 t^2} \exp\left(-\frac{r^2 + r'^2}{4\nu t}\right) \frac{I_1(rr'/2\nu t)}{rr'/2\nu t}. \tag{4.7}$$

Since

$$\lim_{x \to 0} I_1(x)/x = \tfrac{1}{2},$$

the corresponding result for a line impulse of angular momentum is

$$v(r,t) = \frac{r\Omega}{16\pi\nu^2 t^2} \exp\left(-\frac{r^2}{4\nu t}\right). \tag{4.8}$$

A plot of (4.8) with v, r, and t made non-dimensional in an appropriate manner is given in figure 1.†

† The equation $v = (-Ar/2\nu t^2) \exp(-r^2/4\nu t)$ was first given by Taylor (1918) as a particular solution of the momentum equation (3.10 *b*) which could be used to represent 'a small eddy'. Taylor then compared the decay time of this eddy with measurements of the decay of turbulence behind a grid. In the present paper, the energy field associated with this particular velocity distribution is determined.

We now consider some properties of the line impulse.†

(i) Since there are no solid boundaries, the total angular momentum of the fluid (obtained by substituting (4.8) in (4.6)) is independent of time and equal to Ω.

(ii) The over-all kinetic energy of the fluid,

$$I_k(t) \equiv 2\pi \int_0^\infty e_k r\,dr \equiv \pi\rho \int_0^\infty v^2 r\,dr, \tag{4.9}$$

is a function of time, since integrating (4.9) gives

$$I_k(t) = \rho\Omega^2/128\pi\nu^2t^2. \tag{4.10}$$

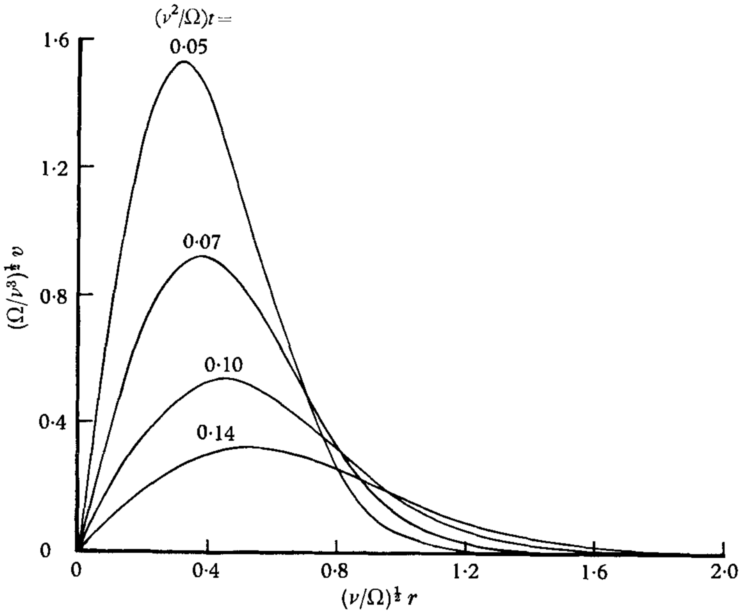

FIGURE 1. Typical profiles showing the decay and spread of the velocity field, v, of a line impulse of angular momentum created at $t = 0$.

Here e_k is the kinetic energy per unit mass. As $t \to 0$, $I_k \to \infty$, i.e. the kinetic energy of the impulse becomes infinite since the momentum remains finite as the mass of fluid in motion approaches zero at time equal to zero. This is clearly a physically unrealizable situation, and will have an important effect on the solution of the energy equation.

(iii) The viscous work, i.e. the rate of increase of energy per unit volume due to the action of shear forces, is (cf. equation (3.10*c*))

$$W(r,t) \equiv \frac{\partial e_k}{\partial t} + \Phi = \frac{\partial}{\partial t}\left(\frac{\rho r^2\omega^2}{2}\right) + \mu r^2\left(\frac{\partial\omega}{\partial r}\right)^2, \tag{4.11}$$

† It may be permissible to point out that (contrary to the case of line and cylindrical heat sources) the cylindrical impulse is not obtained by the superposition of a ring of line impulses since in the latter case the velocity changes direction at $r = r'$.

which for the line impulse becomes

$$W(r,t) = \frac{2\rho\Omega^2}{(16\pi)^2\nu^3t^4}\left[\left(\frac{r^2}{2\nu t}\right)^2 - 2\left(\frac{r^2}{2\nu t}\right)\right]\exp\left(-\frac{r^2}{2\nu t}\right). \tag{4.12}$$

Since this viscous work can cause only a redistribution of energy between fluid elements, we must have

$$2\pi\int_0^\infty W(r,t)\,r\,dr = 0, \tag{4.13}$$

a relationship satisfied by (4.12).

5. Solution of the energy equation for the line impulse

The handling of the unsteady pressure term in the energy equation was discussed in §3 (cf. equation (3.9)). For the velocity field of the line impulse (4.8), we set $a = \infty$ and obtain

$$\frac{1}{\rho}\frac{\partial p}{\partial t} = \frac{\Omega^2}{(16\pi)^2\nu^3t^4}\left(3 - \frac{r^2}{2\nu t}\right)\exp\left(-\frac{r^2}{2\nu t}\right). \tag{5.1}$$

Substituting (4.8) and (5.1) in (3.11) gives

$$\frac{\partial \Xi}{\partial t} - \frac{\nu}{\sigma r}\frac{\partial}{\partial r}\left(r\frac{\partial \Xi}{\partial r}\right) = \frac{\Omega^2}{(16\pi)^2\nu^3}Q(r,t;\,\nu,\sigma), \tag{5.2}$$

where

$$Q(r,t;\,\nu,\sigma) \equiv \frac{1}{t^4}\exp\left(-\frac{r^2}{2\nu t}\right)\left\{2\left(1-\frac{1}{\sigma}\right)\left(\frac{r^2}{2\nu t}\right)^2 + \left[\frac{2}{\sigma} - 4\left(1-\frac{1}{\sigma}\right)\right]\frac{r^2}{2\nu t} - \frac{2}{\sigma} + \left(3 - \frac{r^2}{2\nu t}\right)\delta_{fg}\right\}. \tag{5.2a}$$

Equation (5.2) is of the form of an inhomogeneous, unsteady diffusion equation for the total energy (enthalpy) Ξ in terms of the known distribution of total energy (enthalpy) sources Q. The variable Ξ has been defined (3.11a) such that the boundary condition at infinity is $\Xi(\infty, t) = 0$.

For the liquid case, the energy source term is due only to viscous work and, consequently, equation (5.2a) satisfies the conservation of energy relationship (cf. equation (4.13))

$$2\pi\int_0^\infty Q[\text{liquid}]\,r\,dr = 0. \tag{5.3}$$

In addition, in the absence of heat conduction ($\sigma = \infty$), equation (5.2a) reduces to (4.12), that is to

$$\{\rho\Omega^2/(16\pi)^2\nu^3\}\,Q[\text{liquid},\,\sigma = \infty] = W.$$

For the case of a gas, (5.2a) represents a total enthalpy source field and contains, in addition to the viscous-work terms, a term representing the increase in enthalpy due to fluid element contraction.The distribution of Q for liquids and gases is shown in figure 2 as a function of the similarity parameter $\theta \equiv r^2/2\nu t$, for several values of Prandtl number, σ. For a particular value of σ, the shape of $Q(\theta, t)$ is independent of time, but its magnitude at a specific value of θ decreases as t^4.

Setting

$$\Xi \equiv [\Omega^2/(16\pi)^2 \nu^3] (\Xi_1 + \Xi_2), \tag{5.3a}$$

a formal solution of (5.2) is given by

$$\Xi_1(r,t) = \int_\eta^t dt' \int_0^\infty Q(r',t')\, G(r,t;\, r',t')\, 2\pi r'\, dr', \tag{5.3b}$$

$$\Xi_2(r,t) = \int_0^\infty [\Omega^2/(16\pi)^2 \nu^3]^{-1}\, \Xi(r',\eta)\, G(r,t;\, r',\eta)\, 2\pi r'\, dr', \tag{5.3c}$$

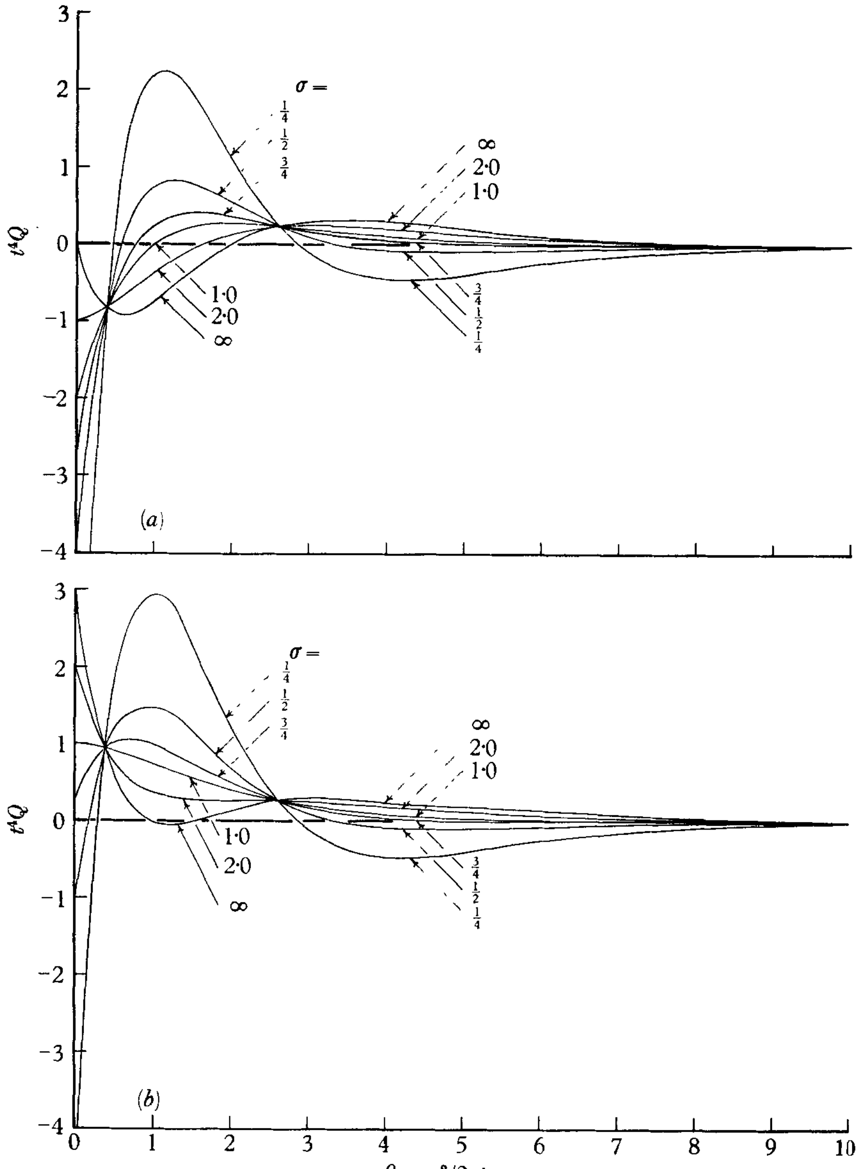

FIGURE 2. Distribution of source strength Q versus the similarity parameter θ for several values of Prandtl number σ. (*a*) Liquid, (*b*) gas.

where G is the Green's function for an instantaneous, cylindrical source (Carslaw & Jaeger 1959, p. 259) satisfying (5.2), namely

$$G(r,t;r',t') \equiv \frac{\sigma}{4\pi\nu}\exp\left[-\frac{\sigma(r^2+r'^2)}{4\nu(t-t')}\right] I_0\left[\frac{\sigma r r'}{2\nu(t-t')}\right]. \tag{5.3d}$$

In (5.3a), Ξ_1 gives the contribution to the energy field of the source distribution $Q(r,t)$, and Ξ_2 gives the contribution of the initial condition $\Xi(r,\eta)$. The singularity in Q at $t=0$ reflects the previously discussed (§4.2) singularity in the kinetic energy at $t=0$. Consequently, a parameter η (having the dimensions of time) has been introduced in (5.3b), and we will look for limiting forms of the solutions of the energy equation for $\eta/t \ll 1$.

After substituting (5.2a) and (5.3d) into (5.3b), the resulting expression for Ξ_1 may be integrated in closed form. The procedure, however, is too lengthy for presentation here; the interested reader may follow the details of the integration in Sibulkin (1960). Briefly, (5.3b) can be integrated with respect to r (using Erdelyi *et al.* 1954), in terms of a finite hypergeometric series. The remaining integration with respect to t can be carried out in terms of elementary functions and the exponential integral $\mathrm{EI}(x)$. For $x \neq 0$, $\mathrm{EI}(x)$ is defined (in terms of the Cauchy principal value) by†

$$\mathrm{EI}(x) \equiv \mathrm{P}\int_x^\infty (e^{-t}/t)\,dt, \tag{5.4a}$$

and is related to previously tabulated functions (Jahnke & Emde 1945, p. 6) by

$$\left.\begin{aligned}\mathrm{EI}(x) &= -\mathrm{Ei}(-x) \quad \text{for} \quad x>0,\\ \mathrm{EI}(x) &= -\overline{\mathrm{Ei}}(-x) \quad \text{for} \quad x<0.\end{aligned}\right\} \tag{5.4b}$$

After defining the variables

$$\theta \equiv \frac{r^2}{2\nu t}, \quad a = \frac{(\sigma-2)\,\theta}{2}, \quad \epsilon = \frac{\sigma(\sigma-2)\,\theta\eta}{2[2t+(\sigma-2)\,\eta]}, \tag{5.5}$$

the solution for the contribution of the source distribution Q to the energy field Ξ is

$$\begin{aligned}\Xi_1(r,t) = {} & \sum_{j=0}^{3}\left[\left(\sum_{i=1}^{10}\alpha_i\beta_i\zeta_{i,j}\right)\left(\frac{2}{\sigma\theta}\right)^j \sum_{k=0}^{j}(-1)^k\frac{k!}{(j-k)!}\,(a^{j-k}e^a-\epsilon^{j-k}e^\epsilon)\right]\\ & +\sum_{j=-3}^{-1}\left[\left(\sum_{i=1}^{10}\alpha_i\beta_i\zeta_{i,j}\right)\left(\frac{2}{\sigma\theta}\right)^j\frac{1}{(-j-1)!}\right.\\ & \left.\times\left\{\mathrm{EI}(-\epsilon)-\mathrm{EI}(-a)+\sum_{k=j}^{-1}[(\epsilon^{k+1}-1)\,e^\epsilon-(a^{k+1}-1)\,e^a]\right\}\right],\end{aligned} \tag{5.6}$$

where $(\alpha\beta)_i$ and $\zeta_{i,j}$ are tabulated in the Appendix.

6. Energy field in a liquid

In §3, we defined a (perfect) liquid by $\rho = \text{const.}$ and $c = \text{const.}$ We also introduced the symbols Ξ and δ_{fg} which for a liquid have the meaning

† This definition of $\mathrm{EI}(x)$ and its relation to $\mathrm{Ei}(x)$ and $\overline{\mathrm{Ei}}(x)$ is given in an unpublished note by A. Farnell, Convair Scientific Research Laboratory, San Diego.

$\Xi = E - e_\infty$, $\delta_{fg} = 0$. As the initial condition for the line impulse in a liquid, we set the total energy E equal to a constant throughout the fluid by assuming

$$E(r, \eta) \equiv e_\infty. \tag{6.1}$$

Then $\Xi(r, \eta) = 0$ which, by (5.3*c*), makes $\Xi_2(r, t) = 0$ and reduces (5.3*a*) to

$$\Xi = [\Omega^2/(16\pi)^2 \nu^3]\, \Xi_1, \tag{6.2}$$

where the complete solution for Ξ_1 is given by (5.6) with the $(\alpha\beta)_i$ terms (cf. Appendix) evaluated for $\delta_{fg} = 0$.

At this point one must consider the significance of the time parameter η. To be precise, (6.2) is the solution for an initial energy distribution (6.1) and an initial velocity distribution given by (4.8) with $t = \eta$, i.e. an initial velocity corresponding to the velocity field of a line impulse of angular momentum at $t = \eta$. As a consequence of the singularity in the kinetic energy of the line impulse at $t = 0$ (cf. §4), one finds that setting $\eta = 0$ in (6.2) yields either $\Xi = +\infty$ or $\Xi = -\infty$ for *all* values of r and t. Consequently, we consider, in §5.1, the approximate form of (6.2) for $\eta/t \ll 1$; and, in §6.2, we investigate the development of (6.2) with η/t for the special case of $\sigma = 1$, for which the explicit expression of (6.2) is greatly simplified.

6.1. *Solution for $\eta/t \ll 1$*

For $\eta/t \ll 1$, the dominant terms in the solution for Ξ_1, (5.6), are those in the series $\sum_{k=j}^{-1} (\epsilon^{k+1} - 1)\, e^{\epsilon}$. For a liquid, $\sum_{i=1}^{10} \alpha_i \beta_i \zeta_{i,-3} = 0$, and the approximate solution for Ξ_1 is given by

$$\Xi_1 \approx \left(\sum_{i=1}^{10} \alpha_i \beta_i \zeta_{i,-2}\right) (\tfrac{1}{2}\sigma\theta)^2 \epsilon^{-1} e^{\epsilon}. \tag{6.3}$$

Carrying out the evaluation of (6.3) for $\eta/t \ll 1$, and using (6.2), yields the energy distribution

$$E(r, t; \eta) - e_\infty \approx \frac{\Omega^2}{2(16\pi)^2 \nu^3} \frac{\sigma(1-2\sigma)(1-\frac{1}{2}\sigma\theta)}{\eta t^2} e^{-\frac{1}{2}\sigma\theta}, \tag{6.4}$$

and defining the non-dimensional variables

$$E^* \equiv \Omega(E - e_\infty)/\nu^3 \quad \text{and} \quad t^* \equiv \nu^2 t/\Omega \tag{6.5}$$

gives the energy distribution in the non-dimensional form

$$E^*(r, t; \eta) \approx \frac{\sigma(1-2\sigma)(1-\frac{1}{2}\sigma\theta)}{2(16\pi)^2 \eta^* (t^*)^2} e^{-\frac{1}{2}\sigma\theta}. \tag{6.4a}$$

In order to eliminate the dependence of E upon η, we define $\bar{e}_k$ (a mean kinetic energy per unit mass at $t = \eta$) by

$$\rho L^2 \bar{e}_k(\eta) \equiv I_k(\eta), \quad L \equiv (\Omega/\nu)^{\frac{1}{2}}, \tag{6.6}$$

where L is a characteristic length. Applying (4.10) and (6.5) to (6.6) yields

$$\bar{e}_k^* = [128\pi(\eta^*)^2]^{-1}. \tag{6.7}$$

Comparing (6.4*a*) and (6.7) one notes that, at a given position in space and time, the total energy increases as the square root of the initial kinetic energy; thus, for $(\eta/t) \to 0$,

$$\frac{E^*}{(\bar{e}_k^*)^{\frac{1}{2}}} \approx \frac{\sigma(1-2\sigma)(1-\frac{1}{2}\sigma\theta)}{32\sqrt{2}\pi^{\frac{3}{2}}(t^*)^2} e^{-\frac{1}{2}\sigma\theta}. \tag{6.8}$$

Equation (6.8) constitutes the solution for the energy field of a line impulse of angular momentum in a liquid. It is interesting to note that, although the shape of the energy input distribution Q is a function of σ (cf. figure 2*a*), only the amplitude of the energy distribution E^* is dependent upon σ while its shape is similar to that of Q for $\sigma = 1$. Since, by (5.3), the over-all rate of energy input into the liquid is zero (relative to the energy level at infinity), the energy distribution (6.8) should, and does, satisfy the condition

$$2\pi \int_0^\infty E^*(r,t)\, r\, dr = 0. \tag{6.9}$$

6.2. *Solution for $\sigma = 1$*

When $\sigma = 1$, the Appendix shows that $\alpha_i = 0$ for $i = 6$ to 10. (This simplification is related to the corresponding simplification for $\sigma = 1$ in the basic energy equation (3.11).) Evaluating the remaining terms in (5.6) from the Appendix for $\sigma = 1$, and making use of (5.4*b*), (6.2), (6.5), and (6.7), gives

$$\frac{E^*(r,t)}{(\bar{e}_k^*)^{\frac{1}{2}}} = \frac{(\eta/t)}{64\sqrt{2}\,\pi^{\frac{3}{2}}(t^*)^2}\left\{(2-2\theta+\tfrac{1}{4}\theta^2)\left[\mathrm{Ei}\left(-\frac{\frac{1}{2}\theta}{[2t/\eta]-1}\right)-\mathrm{Ei}(-\tfrac{1}{2}\theta)\right]e^{-\frac{1}{2}\theta}\right.$$
$$\left.+\left[\frac{1}{(2t/\eta)-1}+3-\left(\frac{2t}{\eta}-1\right)(1-\tfrac{1}{2}\theta)\right]\exp\left[-\frac{\frac{1}{2}\theta}{1-(\eta/2t)}\right]-(3+\tfrac{1}{2}\theta)\,e^{-\theta}\right\}. \tag{6.10}$$

To find E^* for $r = 0$ ($\theta = 0$), we use the series expansion (Jahnke & Emde 1945, pp. 1, 2)

$$\mathrm{Ei}(-x) = 0{\cdot}5772 + \ln x + O(x)$$

to obtain

$$\frac{E^*(0,t)}{(\bar{e}_k^*)^{\frac{1}{2}}} = \frac{1}{32\sqrt{2}\,\pi^{\frac{3}{2}}(t^*)^2}\left[\frac{\eta}{t}\ln\frac{1}{(2t/\eta)-1}-\frac{1-(\eta/t)}{1-(\eta/2t)}\right]. \tag{6.10a}$$

(As a check on the analysis, one can obtain (6.10*a*) directly by setting $r = 0$ and $\sigma = 1$ in (5.4) and integrating the reduced equation for Ξ_1.) The development of E^* as η/t varies from 1 to 0 is shown in figure 3. The result for $\eta/t = 0$ from (6.10) is, of course, identical to (6.8) for $\sigma = 1$.

7. Energy field in a gas

7.1. *Analysis of initial conditions*

In §3 we defined a (perfect) gas by $\rho = p/\mathscr{R}T$ and $c_p = \text{const}$. We also introduced the symbols Ξ and δ_{fg} which for a gas have the meaning $\Xi = H - h_\infty$, $\delta_{fg} = 1$. Once the Ξ field for a gas-flow problem has been determined, the energy field is found from the relationship

$$E = \{H + \tfrac{1}{2}(\gamma-1)\,v^2\}/\gamma. \tag{7.1}$$

It will be assumed that $\gamma > 1$.

As the initial condition for the line impulse in a gas, we again set the energy E equal to a constant throughout the fluid by assuming

$$E(r,\eta) \equiv e_\infty. \tag{7.2}$$

Combining (4.8), (7.1), and (7.2) gives the initial enthalpy distribution

$$\frac{\Xi(r,\eta)}{\Omega^2/(16\pi)^2\nu^3} = -\frac{(\gamma-1)}{\eta^3}\left(\frac{r^2}{2\nu\eta}\right)\exp\left(-\frac{r^2}{2\nu\eta}\right). \tag{7.3}$$

After substituting (7.3) into (5.3*c*), the resulting equation for Ξ_2 can be integrated (Sibulkin 1960) to obtain

$$\Xi_2 = \frac{2(\gamma-1)\,\sigma}{\eta^2}\left\{\frac{(t-\eta)\,[2t+(\sigma-2)\,\eta]+\sigma^2\eta r^2/4\nu}{[2t+(\sigma-2)\,\eta]^3}\right\}\exp\left\{-\frac{\sigma r^2}{2\nu}\frac{1}{[2t+(\sigma-2)\,\eta]}\right\}, \tag{7.4}$$

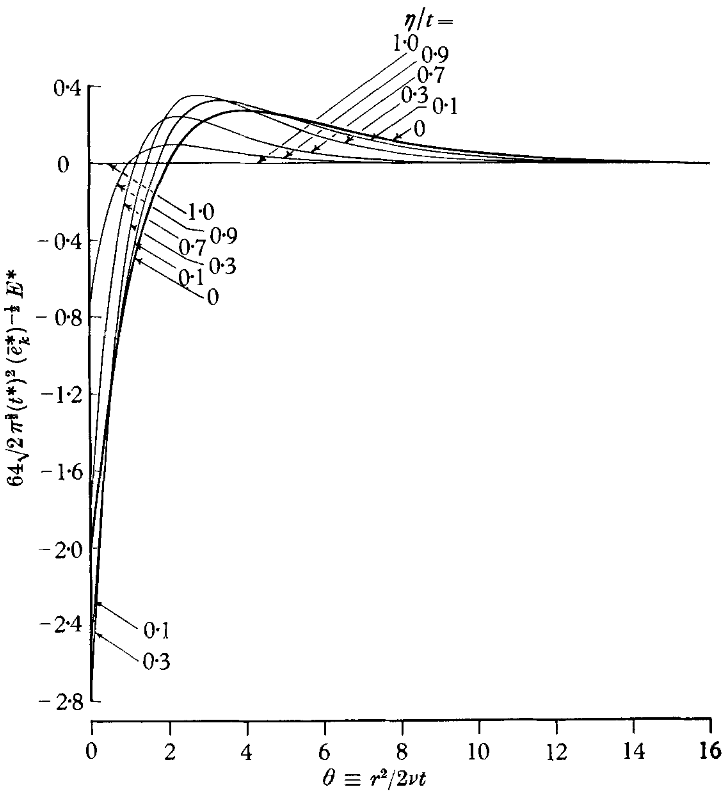

FIGURE 3. Development of the energy field, E^*, in a liquid having a Prandtl number $\sigma = 1$ as a function of the parameter η/t. For fixed time t, the portion of the ordinate in brackets is constant; the solution for the line impulse is the curve at the limit $\eta/t = 0$.

which, when combined with (5.6) in (5.3*a*), gives the general solution for the total enthalpy field, Ξ ($= H - h_\infty$), in a gas. The approximate form of the solution for $\eta/t \ll 1$ and the solution for $\sigma = 1$ will be considered in §§ 7.2 and 7.3.

For either a liquid or a gas, (7.2) sets the initial local-energy-perturbation, $(E-e_\infty)$, equal to zero. In addition, for a liquid, since $\rho =$ const., condition (7.2) is sufficient to make the initial over-all energy-perturbation,

$$I_T = 2\pi\int_0^\infty (\rho E - \rho_\infty e_\infty)\, r\,dr,$$

equal to zero; thereafter, conservation of energy requires that I_T remain equal to zero (cf. (6.9)). The corresponding situation for a gas is more complicated.

Returning, for a moment, to the non-dimensional variables (3.3) used in the preliminary analysis of § 3, and defining

$$I_T^*(t) \equiv 2\pi \int_0^\infty \frac{(\rho E - \overline{\rho e})}{\mathcal{M}\overline{\rho e}}\, r\,dr \quad \text{and} \quad E^{**} \equiv \frac{E-\bar{e}}{\frac{1}{2}V^2}, \tag{7.5}$$

we derive the result

$$I_T^* = 2\pi \int_0^\infty (\gamma E^{**} + \rho^* + \gamma \mathcal{M} \rho^* E^{**})\, r\,dr. \tag{7.6}$$

Consistent with our previous restriction to gas flows in the limit $M \to 0$, (7.6) reduces to

$$I_T^* = \gamma I_E^* + I_\rho^*, \tag{7.7a}$$

after defining

$$I_E^*(t) \equiv 2\pi \int_0^\infty E^{**}(r,t)\, r\,dr \quad \text{and} \quad I_\rho^*(t) \equiv 2\pi \int_0^\infty \rho^*(r,t)\, r\,dr, \tag{7.7b}$$

where I_ρ^* is the over-all mass-perturbation. Applying the first law of thermodynamics, between time $t' = \eta$ and $t' = t$, to an open system bounded by radius R (see, for example, Keenan 1941, pp. 32ff.) gives

$$2\pi \int_0^R \rho(r,t)\, E(r,t)\, r\,dr = 2\pi \int_0^R \rho(r,\eta)\, E(r,\eta)\, r\,dr$$
$$+ 2\pi \int_\eta^t \left[E(R,t') + \frac{p(R,t')}{\rho(R,t')} \right] \int_0^R \frac{\partial \rho(r,t')}{\partial t'}\, r\,dr\,dt'. \tag{7.8}$$

The last term in (7.8) expresses the change in energy of the fluid within R due to the energy content of the fluid crossing the boundary R and to the work done on the system in moving this fluid across R. Now if we let $R \to \infty$ and use the integrals defined in (7.5) and (7.7b), (7.8) reduces to

$$I_T^*(t) = I_T^*(\eta) + \gamma[I_\rho^*(t) - I_\rho^*(\eta)]. \tag{7.9}$$

Thus, for a gas, in contrast to the situation for a liquid, the total-energy integral I_T^* is not necessarily constant with time, but depends upon the variation with time of the mass integral I_ρ^*. This is due to the possibility, for a gas, of the fluid 'at infinity' exchanging energy with the interior fluid. In order to apply (7.9) to the solution obtained in the next section, the initial values of I_T^* and I_ρ^* must be determined. However, since (7·2) makes $I_E^*(\eta)$ identically equal to zero, $I_T^*(\eta) = I_\rho^*(\eta)$, and the problem is reduced to the determination of the initial density distribution $\rho^*(r,\eta)$.

The density distribution $\rho^*(r,\eta)$ is given in terms of $p^*(r,\eta)$ and $h^*(r,\eta)$ by the perfect gas law (3.4). If we identify the mean fluid properties, $(\bar{\ })$, used in the general analysis of § 3 with the fluid properties at infinity, $(\ )_\infty$, in the case of the line impulse, $p^*(r,\eta)$ can be found by integrating the radial momentum equation (3.8a) for the velocity distribution (4.8); $h^*(r,\eta)$ follows directly from (7.1) and (7.2). The combined result is

$$\rho^*(r,\eta) = \frac{2\gamma}{V^2} \frac{\Omega^2}{(16\pi)^2 \nu^3 \eta^3} \left(\theta - \frac{1}{\gamma - 1} \right) e^{-\theta}, \tag{7.10}$$

which, when integrated, gives for the initial value of the mass integral the result

$$I^*_\rho(\eta) = \frac{4\pi\gamma(\gamma-2)}{V^2(\gamma-1)}\frac{\Omega^2}{(16\pi)^2\nu^2\eta^2}. \tag{7.11}$$

7.2. *Solution for $\eta/t \ll 1$*

As in the liquid case (cf. § 6.1), the dominant terms in (5.6) for $\eta/t \ll 1$ are those in the series $\sum_{k=j}^{-1}(\epsilon^{k+1}-1)e^{\epsilon}$. For a gas, however, $\sum_{i=1}^{10}\alpha_i\beta_i\zeta_{i,-3} \neq 0$, and, keeping terms of $O(\eta^{-1})$, the approximate solution for Ξ_1 is

$$\Xi_1 \approx \tfrac{1}{2}\sigma\left\{\frac{1}{\eta^2 t}+\frac{1}{\eta t^2}[-\tfrac{5}{2}\sigma+3+\sigma(\sigma-1)\theta]\right\}e^{-\frac{1}{2}\sigma\theta}. \tag{7.12}$$

To the same order of magnitude, the approximate form of (7.4) is

$$\Xi_2 \approx \tfrac{1}{2}(\gamma-1)\,\sigma\left\{\frac{1}{\eta^2 t}+\frac{1}{\eta t^2}[-\sigma+1+\tfrac{1}{4}\sigma^2\theta]\right\}e^{-\frac{1}{2}\sigma\theta}. \tag{7.13}$$

Comparing the magnitude of $H-h_\infty$ given by (7.12) and (7.13) with the magnitude of v^2 given by (4.8) shows that the corresponding approximate form of (7.1) is

$$E-e_\infty \approx (H-h_\infty)/\gamma; \tag{7.14}$$

that is, for $\eta/t \ll 1$, the kinetic energy is much less than the induced increment of thermal energy. Substituting (7.12) and (7.13) into (5.3a) and applying (7.14) gives, for $\gamma \neq 2$,

$$E-e_\infty \approx -\left(\frac{\gamma-2}{\gamma}\right)\frac{\sigma\Omega^2}{2(16\pi)^2\nu^3\eta^2 t}e^{-\frac{1}{2}\sigma\theta}. \tag{7.15a}$$

For $\gamma = 2$, the terms of $O(\eta^{-2})$ cancel identically and, using (6.5) and (6.7), the non-dimensional energy distribution is

$$\frac{E^*}{(\bar{e}^*_k)^{\frac{1}{2}}} \approx \frac{\sigma(1-\frac{3}{4}\sigma)(1-\frac{1}{2}\sigma\theta)}{32\sqrt{2}\pi^{\frac{3}{2}}(t^*)^2}e^{-\frac{1}{2}\sigma\theta}. \tag{7.15b}$$

Equations (7.15) constitute the solution for the energy field of a line impulse of angular momentum in a gas. As such they must satisfy the energy relationships derived in § 7.1. Applying the same considerations that led to (7.14), one can show, for $\eta/t \ll 1$, that $e-e_\infty \approx E-e_\infty$ and that $p^* \ll h^*$, which reduces the gas law (3.4) to $\rho^* \approx -\gamma e^*$. Consequently, the mass integral $I^*_\rho(t)$ may be obtained by integrating (7.15a); the result is

$$I^*_\rho(t) = -\gamma I^*_E(t) = \frac{4\pi(\gamma-2)}{V^2}\frac{\Omega^2}{(16\pi)^2\nu^2\eta^2}. \tag{7.16}$$

Combining (7.9), (7.11), and (7.16) gives

$$I^*_\rho(t)-I^*_\rho(\eta) = -(1/\gamma)\,I^*_\rho(\eta) \quad\text{and}\quad I^*_T(t)-I^*_T(\eta) = -I^*_\rho(\eta),† \tag{7.17}$$

which shows that,

$$\left.\begin{array}{lll} \text{for } \gamma < 2, & I^*_\rho(t) > I^*_\rho(\eta) \text{ and } I^*_T(t) > I^*_T(\eta); \\ \text{for } \gamma = 2, & I^*_\rho(t) = I^*_\rho(\eta) \text{ and } I^*_T(t) = I^*_T(\eta); \\ \text{for } \gamma > 2, & I^*_\rho(t) < I^*_\rho(\eta) \text{ and } I^*_T(t) < I^*_T(\eta). \end{array}\right\} \tag{7.18}$$

† Since $I^*_T(t) = 0$ and $I^*_T(\eta) = I^*_\rho(\eta)$, this equation also shows that the solution for the energy field for $\eta/t \ll 1$, (7.15), satisfies the first law of thermodynamics expressed in the form (7.9).

That is, for $\gamma < 2$, there is a net compression of the fluid set in motion by the line impulse, and the work accompanying this compression causes the total energy of the line impulse to increase with time. For $\gamma > 2$, the opposite occurs; for $\gamma = 2$, the energy of the line impulse remains constant, and this is the physical explanation for the cancellation of the higher-order terms in (7.12) and (7.13) for $\gamma = 2$.

Since the net effect of compression is zero for the $\gamma = 2$ gas, it is not surprising that the energy field of the line impulse in that case, (7.15*b*), has the same form as the energy field for the line impulse in a liquid, (6.8). However, although the *net* effect of compression is zero for the $\gamma = 2$ gas, there is still a *local* increase in enthalpy due to compression (cf. equation (5.1)) for $\theta < 3$ and a corresponding decrease for $\theta > 3$. To show the effect of the local density changes more fully, the development of the energy field in a $\sigma = 1, \gamma = 2$ gas will be considered in the next section.

7.3. *Solution for* $\sigma = 1, \gamma = 2$

As was the case for the liquid (cf. §6.2), the evaluation of (5.6) is considerably simpler when the Prandtl number $\sigma = 1$. The result is (Sibulkin 1960)

$$\frac{E^*}{(\bar{e}_k^*)^{\frac{1}{2}}} = \frac{1}{64\sqrt{2}\,\pi^{\frac{3}{2}}(t^*)^2}\left\{\frac{1}{2}\left[1-\tfrac{1}{2}\theta+\frac{(8-6\theta)(\eta/t)-(6-3\theta)(\eta/t)^2+(1-\frac{1}{2}\theta)(\eta/t)^3}{[2-(\eta/t)]^3}\right]\right.$$
$$\left.\times\exp\left[-\frac{\frac{1}{2}\theta}{1-(\eta/2t)}\right]+\frac{2\eta}{t}(\theta-1)\,e^{-\theta}\right\}. \quad (7.19)$$

The development of E^* as η/t varies from 1 to 0 is shown in figure 4. The result for $\eta/t = 0$ from (7.19) is, of course, identical to (7.15*b*) for $\sigma = 1$. Comparing figures 3 and 4, it can be seen that, during the early stages of development of the line impulse, say for $\eta/t > 0{\cdot}5$, the energy distributions are not too different in that E^* is negative at $\theta = 0$ for both the liquid and the gas. Later on, as $\eta/t \rightarrow 0$ however, the local effects of compressibility (as discussed in §7.2) cause E^* to become positive at $\theta = 0$ and continue to cause E^* to be negative at values of $\theta > 3$.

7.4. *Discussion of results*

The results obtained in the preceding sections lead to the following qualitative description of the history of what we have defined as a *line impulse of angular momentum*, (4.6).

At the time the line impulse originates, $t = \eta$, the fluid is nearly at rest except in a region near the axis where, for $r^2/2\nu\eta < O(10)$, the circumferential velocity v rises sharply to a maximum before returning to zero at the axis. Corresponding to this velocity maximum, there is a temperature minimum (relative to the temperature of the fluid at infinity) such that the total energy—kinetic plus thermal—is constant throughout the fluid. And, as a consequence of the velocity field, there is a pressure minimum at the axis.†

† It is suggested that the interested reader sketch the profiles at $t = \eta$ for reference during the remainder of this discussion.

As the line impulse decays for $t > \eta$, a 'shear wave' radiates from the axis in the sense that the radius of the velocity maximum increases with time while its magnitude decreases (figure 1). Simultaneously, a 'thermal wave' and a 'pressure wave' propagate radially. If we fix our attention on a fluid element initially at rest at a radius r' (which in the limit $\eta \to 0$ includes all fluid elements), the passage of the shear wave sets the fluid elements into circular† motion about the axis of the line impulse with a velocity which rises to a maximum and then

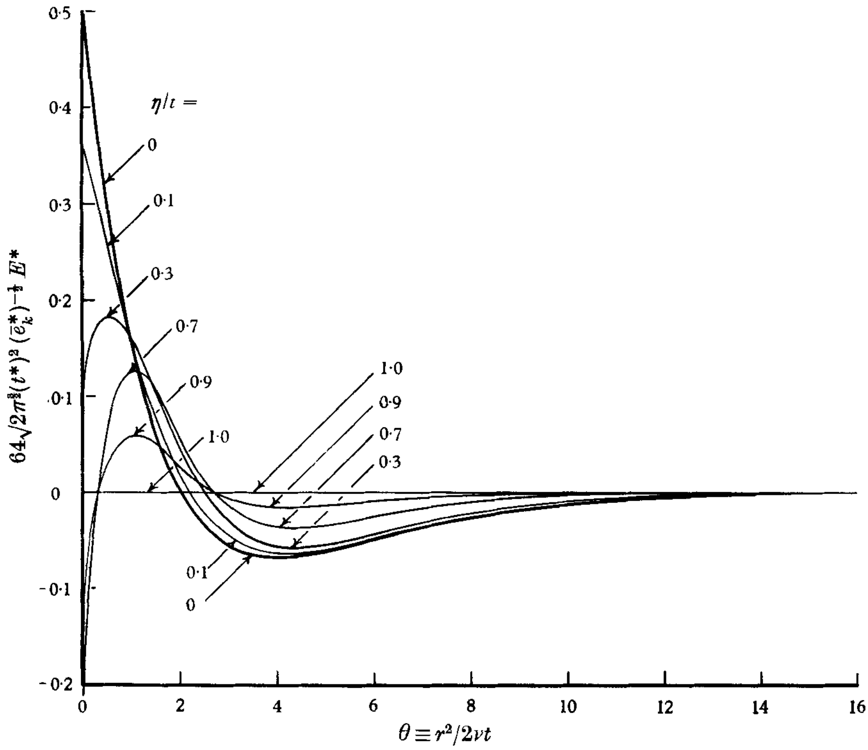

FIGURE 4. Development of the energy field E^* in a gas having a Prandtl number $\sigma = 1$ and a ratio of specific heats $\gamma = 2$ as a function of the parameter η/t. For a fixed time t, the portion of the ordinate in brackets is constant; the solution for the line impulse is the curve at the limit $\eta/t = 0$.

decays to zero as $t \to \infty$. Thus the initial tendency of the shear wave is to increase the energy of the element by increasing its kinetic energy. On the other hand, the passage of the thermal wave tends to decrease the energy of the element. For Prandtl numbers σ greater than one, the kinematic viscosity is greater than the thermal diffusivity and the shear wave spreads more rapidly than the thermal wave causing the total energy of the element at r' to increase at first; for $\sigma < 1$, the reverse occurs. These effects can be traced on figure 2 by noting that, at a fixed radius $r = r'$, θ decreases as t increases.

† For a liquid the streamlines are circles; for a gas, subject to the limitation $M \ll 1$ assumed in this paper, the streamlines are perturbed circles in that $r(t) \to r'$ as $M \to 0$, as a consequence of (3.6).

In addition to the effects of shear and heat conduction described above, the passage of the pressure wave causes the pressure at r' to decrease at first and then tends to return the pressure to its initial value; and, for a gas, the work of compression accompanying this pressure variation first decreases and then increases the energy of the element. The over-all effect of compressibility depends upon the value of γ for the gas (as discussed in §7.1) in such a way that the total energy of the fluid set in motion by the line impulse increases with time for $\gamma < 2$ and decreases for $\gamma > 2$ (7.15*a*). For $\gamma = 2$, the total energy of the gas remains constant, and the solution for the energy field in this case, (7.15*b*), has the same form as the solution for the liquid (6.8). These solutions show that for $t/\eta \gg 1$, $E^* - e_\infty \approx 0$ throughout the fluid for $\sigma = \frac{1}{2}$ in the case of a liquid and for $\sigma = \frac{4}{3}$ in the case of the gas. The differences between the liquid and the $\gamma = 2$ gas are due to the local effects of compressibility, and are further illustrated by the differences in the development of the line impulse with η/t as shown in figures 3 and 4 and discussed in §7.3.

Appendix

The values of $(\alpha\beta)_i$ and $\zeta_{i,j}$ are tabulated below.

	$(\alpha\beta)_i \,.\, 4\sigma\theta t^3 \exp(\frac{1}{2}\sigma\theta)$
$i = 1$	$(-2+3\sigma\delta_{fg})(\sigma-2)^2$
2	$2[1-2(\sigma-1)-\frac{1}{2}\sigma\delta_{fg}](\sigma-2)^2$
3	$-4[1-2(\sigma-1)-\frac{1}{2}\sigma\delta_{fg}](\sigma-2)$
4	$[1-2(\sigma-1)-\frac{1}{2}\sigma\delta_{fg}]\,\sigma^2(\sigma-2)\,\theta$
5	$4(\sigma-1)(\sigma-2)^2$
6	$-16(\sigma-1)(\sigma-2)$
7	$16(\sigma-1)$
8	$4\sigma^2(\sigma-1)(\sigma-2)$
9	$-8\sigma^2(\sigma-1)\,\theta$
10	$\frac{1}{2}\sigma^4(\sigma-1)\,\theta^2$

	$\zeta_{i,j}$						
$j =$	-3	-2	-1	0	1	2	3
$i = 1$	1	-2	1	0	0	0	0
2	1	-3	$+3$	-1	0	0	0
3	0	1	-2	1	0	0	0
4	0	1	-3	$+3$	-1	0	0
5	1	-4	$+6$	-4	$+1$	0	0
6	0	1	-3	$+3$	-1	0	0
7	0	0	1	-2	1	0	0
8	0	1	-4	$+6$	-4	1	0
9	0	0	1	-3	$+3$	-1	0
10	0	0	1	-4	$+6$	-4	1

REFERENCES

CARSLAW, H. S. & JAEGER, J. C. 1959 *Conduction of Heat in Solids*, 2nd edn. Oxford University Press.

ERDELYI, A., MAGNUS, W., OBERHETTINGER, F. & TRICOMI, F. G. 1954 *Tables of Integral Transforms*, Vol. 1. New York: McGraw-Hill.

HOWARTH, L. (ed.) 1953 *Modern Developments in Fluid Dynamics, High Speed Flow*, Vol. 1. Oxford University Press.

JAHNKE, E. & EMDE, F. 1945 *Tables of Functions*. New York: Dover.

KEENAN, J. H. 1941 *Thermodynamics*. New York: John Wiley.

MCLEOD, A. R. 1922 The unsteady motion produced in a uniformly rotating cylinder of water by a sudden change in the angular velocity of the boundary. *Phil. Mag.* **44**, 1–14.

PAI, S-I. 1956 *Viscous Flow Theory*. I. *Laminar Flow*. Princeton: Van Nostrand.

SIBULKIN, M. 1960 Unsteady, viscous, circular flow. Part I. The line impulse of angular momentum. *Convair Sci. Res. Lab., San Diego, Res. Note* no. 39.

TAYLOR, G. I. 1918 On the dissipation of eddies. *Adv. Comm. Aero., Lond., Rep. and Mem.* no. 598. (Available in Batchelor, G. K. (ed.) 1960, *The Scientific Papers of G. I. Taylor*, Vol. II, p. 96. Cambridge University Press.)